Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

sam.nitk.ac.in nitksam@gmail.com

Advanced Linear Algebra (MA 409)
Problem Sheet - 15

Determinants of order 2

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1. Label the following statements as true or false.
   (a) The function $\det : M_{2\times 2}(F) \to F$ is a linear transformation.
   (b) The determinant of a 2×2 matrix is a linear function of each row of the matrix when the other row is held fixed.
   (c) If $A \in M_{2\times 2}(F)$ and $\det(A) = 0$, then A is invertible.
   (d) If u and v are vectors in $\mathbb{R}^2$ emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is
   $$\det \begin{pmatrix} u \\ v \end{pmatrix}.$$
   (e) A coordinate system is right-handed if and only if its orientation equals 1.

2. Compute the determinants of the following matrices in $M_{2\times 2}(\mathbb{R})$.

   a) $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$ b) $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

3. Compute the determinants of the following matrices in $M_{2\times 2}(\mathbb{C})$.

   a) $\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$ b) $\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$ c) $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

4. For each of the following pairs of vectors u and v in $\mathbb{R}^2$, compute the area of the parallelogram determined by u and v.
   (a) $u = (3, -2)$ and $v = (2, 5)$
   (b) $u = (1, 3)$ and $v = (-3, 1)$
   (c) $u = (4, -1)$ and $v = (-6, -2)$
   (d) $u = (3, 4)$ and $v = (2, -6)$

5. Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.

6. Prove that if the two columns of $A \in M_{2\times 2}(F)$ are identical, then $\det(A) = 0$.

7. Prove that $\det(A^t) = \det(A)$ for any $A \in M_{2\times 2}(F)$.

8. Prove that if $A \in M_{2\times 2}(F)$ is upper triangular, then $\det(A)$ equals the product of the diagonal entries of A.

9. Prove that $\det(AB) = \det(A) \cdot \det(B)$ for any $A, B \in M_{2\times 2}(F)$.

10. The **classical adjoint** of a 2×2 matrix $A \in M_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Prove that

(a) $CA = AC = [\det(A)]I$.

(b) $\det(C) = \det(A)$.

(c) The classical adjoint of A^t is C^t.

(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

11. Let $\delta : M_{2\times 2}(F) \to F$ be a function with the following three properties.

(i) δ is a linear function of each row of the matrix when the other row is held fixed.

(ii) If the two rows of $A \in M_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.

(a) If I is the 2×2 identity matrix, then $\delta(I) = 1$.

Prove that $\delta(A) = \det(A)$ for all $A \in M_{2\times 2}(F)$.

12. Let $\{u, v\}$ be an ordered basis for $\mathbb{R}^2$. Prove that

$$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ forms a right-handed coordinate system.

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